

Quantitative study of the transverse correlation of soft gluons in high energy QCD

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ABSTRACT: We examine both analytically and numerically the validity of factorization for the double dipole scattering amplitude $T^{(2)}$ which appears on the right hand side of the BK-JIMWLK equation. We demonstrate that, if one uses a dilute object (e.g., a proton in DIS) as the initial condition, the correlation in the transverse plane induced by the leading order BFKL evolution is generally strong, resulting in a violation of the mean field approximation $T^{(2)} \approx TT$ even at zero impact parameter by a factor ranging from 1.5 to $\mathcal{O}(10)$ depending on the relative size of the scatterers and rapidity. This suggests that, within the experimentally accessible energy interval, the transverse correlation can significantly affect the nonlinear evolution of the dipole scattering amplitude. It also suggests that the nonlinear effects may set in earlier, already in the weak scattering regime. In the case of the simulation with a running coupling, the violation of factorization is somewhat milder, but still noticeable.

KEYWORDS: Deep Inelastic Scattering, Hadronic Colliders, QCD.

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1. Introduction

High energy scattering near the unitarity limit is a delicate problem which deserves intense theoretical efforts in view of its phenomenological importance at hadron colliders. There is a clear goal of including nonlinear, saturation effects due to the high density of gluons into the energy evolution of scattering amplitudes, but a precise determination of when and how these effects should be treated is subject to various uncertainties depending on the process of interest. The problem appears to somewhat simplify if one considers scattering of a small object (e.g., a photon at high virtuality in DIS) off a very heavy nucleus where saturation is important already at relatively low energy. For such a process the Balitsky-Kovchegov (BK) equation [1, 2] is the most commonly studied equation which provides a concrete scenario for an approach towards unitarity,

$$\begin{aligned}
 \partial_Y T_Y(x, y) &= \frac{\bar{\alpha}_s}{2\pi} \int d^2z \mathcal{M}(x, y, z) \times \\
 &\quad \times \left\{ -T_Y(x, y) + T_Y(x, z) + T_Y(z, y) - T_Y(x, z)T_Y(z, y) \right\}, \\
 \mathcal{M}(x, y, z) &\equiv \frac{(x - y)^2}{(x - z)^2(z - y)^2}, \quad \bar{\alpha}_s \equiv \frac{\alpha_s N_c}{\pi}.
 \end{aligned}
 \tag{1.1}$$

Here $T_Y(x, y)$ is the forward amplitude of a dipole of size $|x - y|$ at rapidity Y . The first three terms on the right hand side contain the BFKL physics [3, 4] while the last term $\sim TT$ ensures that the amplitude saturates the black disc limit $T \rightarrow 1$ which is a fixed

point of the equation. Being a closed equation, (1.1) is amenable to both analytical and numerical approaches, and the properties of the solution as well as their phenomenological consequences have been discussed extensively over the past several years (see, reviews [5, 6] and references therein).

However, it is not often emphasized that the BK equation is a mean field approximation to a more general equation, namely, the B-JIMWLK equation [1, 7–10]

$$\partial_Y T_Y(x, y) = \frac{\bar{\alpha}_s}{2\pi} \int d^2 z \mathcal{M}(x, y, z) \times \left\{ -T_Y(x, y) + T_Y(x, z) + T_Y(z, y) - \langle T_Y(x, z) T_Y(z, y) \rangle \right\}, \quad (1.2)$$

nor is the validity of this approximation fully appreciated. Here the brackets $\langle \dots \rangle$ denote averaging over the target configurations. The difference between these two equations is usually considered to be minor: Although the former obviously discards any kind of existing correlations in the target wavefunction, this would be justified for a large nucleus at low rapidity (see, however, [11]). The subsequent quantum evolution then generates correlations which vanish in the large N_c limit,

$$\langle TT \rangle \approx \langle T \rangle \langle T \rangle + \mathcal{O} \left(\frac{1}{N_c^2} \right). \quad (1.3)$$

Indeed, the only existing numerical simulation of the B-JIMWLK equation [12] starting from uncorrelated initial conditions shows little difference from the corresponding solution to the BK equation.

The purpose of this work is to demonstrate that the factorization (1.3) is violated when one considers a dilute target consisting of a few partons (e.g., a proton) instead of a heavy nucleus as the initial condition. Of course, there is *a priori* no reason to expect that factorization should work in this case, but there has not been any quantitative study of the degree of its violation either. For a dilute target, a significant part of the rapidity evolution in realistic experiments is in the linear BFKL regime where the amplitude is rapidly growing but still much less than unity, whereas saturation is considered to be relevant only in the late stages of the evolution.¹ The fluctuations and correlations developed in the linear regime are so strong that the initial condition that should be used for the nonlinear evolution equations is a highly nontrivial system of gluons for which the difference between (1.1) and (1.2) may turn out to be crucial, especially for phenomenology. Specifically, in the framework of the QCD dipole model ref. [13] found a power-law correlation in the double scattering amplitude²

$$\langle T(x, z) T(w, y) \rangle \propto \frac{1}{|z - w|^\gamma}, \quad (1.4)$$

under the condition that the distance between the two dipoles are much larger than their sizes, $|z - w| \gg |x - z|, |w - y|$. (γ is a positive, calculable number related to the

¹However, we have found some evidence that nonlinear effects might set in earlier due to the correlation. See the discussion in section 3.2.

²See also [14], though there seem to be disagreements in the results.

anomalous dimension.) In the exemplary cases studied in [13], this power-law always leads to a parametrically large ratio

$$R \equiv \frac{\langle T(x, z)T(w, y) \rangle}{\langle T(x, z) \rangle \langle T(w, y) \rangle} \gg 1. \quad (1.5)$$

Due to a technical reason, in [13] it was not possible to take the interesting limit $w \rightarrow z$ to evaluate R for the ‘BK configuration’, although it was tantalizing to conclude from (1.4) that that the correlation would become even larger in this case. Here we circumvent this difficulty and present an analytical insight into the behavior of R as a function of the initial dipole sizes.

However, analytical calculations are often quite difficult, and one can usually only deal with special configurations which are set by hand. Besides, for our purpose it is important to know the actual numerical value of T and $\langle T^2 \rangle$ to make sure that one evaluates R in a regime where the nonlinear corrections just start to be important. We will therefore also perform a Monte Carlo (MC) simulation of the QCD dipole model [15] which contains the exact leading order BFKL dynamics. In this framework one generates a cascade of dipoles keeping track of their sizes and positions in the transverse plane. Calculations of $\langle T^k \rangle$ for any k , hence R , are completely straightforward for arbitrary configurations. We then compare the numerical results with analytic expectations and find that they agree satisfactorily. For zero impact parameter we find that R is much larger than 1 when the ratio of the projectile and target sizes is either small or large. The minimum value for R is attained when the projectile and target are of similar size, and in this case the value of R is around 1.5. This suggests that, in the leading logarithmic approximation on which both the BK equation and the dipole model are based, the replacement $\langle TT \rangle \rightarrow \langle T \rangle^2$ is not valid for a proton target especially for a small dipole projectile (or in the high- Q^2 region of DIS), although it might be safe to do so for a nucleus target. In the former case one should rather use the B-JIMWLK equation with a strongly correlated initial condition, whose asymptotic solution can be different from that of the BK equation.

The fact that one finds large correlations in the leading order evolution for a dilute system is consistent with the early studies on fluctuations in [16, 17]. In [16] it was found that $\langle T^k \rangle \sim (k!)^2$ (or rather $\langle T^k \rangle \sim k! \cdot (k + 3)!$) at zero impact parameter. This implies that, for any $m \leq k$,

$$\frac{\langle T^k \rangle}{\langle T^{k-m} \rangle \langle T^m \rangle} \sim \left(\frac{k!}{(k-m)! m!} \right)^2 = \binom{k}{m}^2 \gg 1. \quad (1.6)$$

Note, however, that the definition of $\langle T^k \rangle$ in (1.6) is different from the one considered in this paper, namely, $\langle T^k \rangle$ appearing in the Balitsky hierarchy whose first equation is (1.2). In (1.6), one evolves the target and the projectile up to some energy, and then calculate the sum of all events in which there are k simultaneous interactions. In our case we rather fix k given dipoles in the transverse plane, and then consider their scattering off some target. Only the latter contains information of the correlation resolved in the transverse plane.

In [18–20] the dipole model has been modified and extended to include various non-leading effects as well as saturation and confinement effects during the evolution. Generally

speaking, these effects tend to reduce the correlation. For example, $\langle T^k \rangle$ as defined in [16] behaves as (for k between 5 and 9) $\langle T^k \rangle / \langle T^{k-1} \rangle \approx 1.2 \cdot k$ once the nonleading effects are included [20]. This implies

$$\frac{\langle T^k \rangle}{\langle T^{k-m} \rangle \langle T^m \rangle} \sim \binom{k}{m}, \quad (1.7)$$

and thus the correlation is reduced with respect to (1.6). It should, however, be said that the fluctuations are still very important, and they have for example important consequences on the study of elastic and diffractive scattering in DIS and pp collisions [20]. In this paper we only show some of the preliminary numerical results with the running coupling effect to see if there is a similar suppression of the correlation, while a detailed study of the various additional effects is postponed to a future publication.

The paper is organized as follows. In the next section we present analytical calculations of the double dipole scattering amplitude and the ratio R for the BK configurations mentioned above. In section 3.1 we outline our numerical approach to the calculation of the correlation. The results, including the running coupling case, are then presented in section 3.2 where we also make comparison with the analytical expectations. Finally, in section 4 we summarize our results and raise some open questions.

2. Analytical approach

2.1 The dipole pair density

In the dipole model [15], the degree of the two-body correlation in impact parameter space is encoded in the dipole pair density [21, 22] whose integral representation reads (keeping only the zero conformal spin sector) [23, 24]

$$\begin{aligned} n_Y^{(2)}(x_{01}, x_{a_0 a_1}, x_{b_0 b_1}) &= \int d\gamma d\gamma_a d\gamma_b \frac{1}{2x_{a_0 a_1}^2 x_{b_0 b_1}^2} \int_0^Y dy e^{\chi(\gamma)y + (\chi(\gamma_a) + \chi(\gamma_b))(Y-y)} \\ &\times \int d^2 x_\alpha d^2 x_\beta d^2 x_\gamma E^\gamma(x_{0\gamma}, x_{1\gamma}) E^{\gamma_a}(x_{a_0\alpha}, x_{a_1\alpha}) E^{\gamma_b}(x_{b_0\beta}, x_{b_1\beta}) \\ &\times \int \frac{d^2 x_2 d^2 x_3 d^2 x_4}{x_{23}^2 x_{34}^2 x_{42}^2} E^{1-\gamma}(x_{2\gamma}, x_{3\gamma}) E^{1-\gamma_a}(x_{2\alpha}, x_{4\alpha}) E^{1-\gamma_b}(x_{3\beta}, x_{4\beta}) \end{aligned} \quad (2.1)$$

where $x_{01} = x_0 - x_1$ denotes the coordinate of the parent dipole, and $x_{a_0 a_1} = x_{a_0} - x_{a_1}$ and $x_{b_0 b_1} = x_{b_0} - x_{b_1}$ are those of the child dipoles (see, figure 1). We shall use the letter x for both two-dimensional real vectors and their magnitude. χ is the BFKL eigenvalue

$$\chi(\gamma) = \bar{\alpha}_s (2\psi(1) - \psi(\gamma) - \psi(1 - \gamma)), \quad (2.2)$$

with γ being the anomalous dimension, and E is the eigenfunction of the $SL(2, \mathbb{C})$ group

$$E^\gamma(x_{0\gamma}, x_{1\gamma}) = \left(\frac{x_{01}}{x_{0\gamma} x_{1\gamma}} \right)^{2\gamma}. \quad (2.3)$$

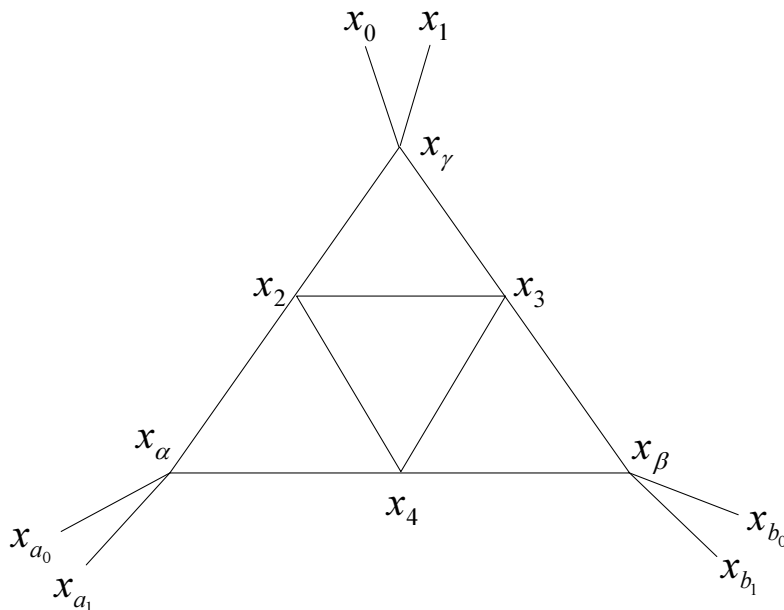


Figure 1: A graphical representation of equation (2.1).

The γ -integrals are along the imaginary axis. With the usual representation $\gamma = \frac{1}{2} + i\nu$, it reads

$$\int d\gamma \equiv \int_{-\infty}^{\infty} d\nu \frac{2\nu^2}{\pi^4}. \quad (2.4)$$

In ref. [13], the multi-dimensional integral in (2.1) has been carried out in the limit

$$x_{ab} = x_a - x_b \equiv \frac{x_{a_0} + x_{a_1}}{2} - \frac{x_{b_0} + x_{b_1}}{2} \gg x_{a_0 a_1}, x_{b_0 b_1}. \quad (2.5)$$

The result shows a power-law correlation between the two child dipoles. In the case of $x_{01} \gg x_{ab}$, ref. [13] found

$$n^{(2)} \sim \left(\frac{x_{01}}{x_{ab}} \right)^{2(2\gamma_a - \gamma)} (n)^2, \quad (2.6)$$

where n is the single dipole density, and γ_a and γ are the saddle point values determined from certain conditions. The breakdown of factorization is carried over to that of the two-dipole scattering amplitude

$$T^{(2)}(x_{a_0 a_1}, x_{b_0 b_1}) \sim \left(\frac{x_{01}}{x_{ab}} \right)^{2(2\gamma_a - \gamma)} T(x_{a_0 a_1}) T(x_{b_0 b_1}) \gg T(x_{a_0 a_1}) T(x_{b_0 b_1}), \quad (2.7)$$

as already noted in the introduction. (From now on we use the notation $T^{(2)}$ in place of $\langle T^2 \rangle$.) On the other hand, the quantity of interest for us is the two dipole scattering amplitude for contiguous dipoles, namely,

$$x_{a_1} = x_{b_1}. \quad (2.8)$$

Although it is not legitimate to extrapolate the result (2.6) to the case $x_{ab} \rightarrow 0$, it does suggest that the correlations would become even larger for such ‘BK configurations’. (The numerical evaluation of this case is presented in section 3.2.) In this section we attempt at an analytical evaluation of $n^{(2)}$ for $x_{a_1} = x_{b_1}$ in certain limits and discuss the behavior of the ratio R defined in (1.5). The result will be confronted with numerical Monte Carlo simulations in the next section.

2.2 Calculation of $n^{(2)}$ for contiguous dipoles

The last line of (2.1) is a known integral whose overall structure is fixed by conformal symmetry. After performing this integral, the last two lines of (2.1) become

$$I \equiv f(\gamma, \gamma_a, \gamma_b) \int d^2x_\alpha d^2x_\beta \left(\frac{x_{a_0c}}{x_{a_0\alpha}x_{c\alpha}} \right)^{2\gamma_a} \left(\frac{x_{b_0c}}{x_{b_0\beta}x_{c\beta}} \right)^{2\gamma_b} \frac{1}{x_{\alpha\beta}^{2(1+\gamma-\gamma_a-\gamma_b)}} \\ \times \int d^2x_\gamma \left(\frac{x_{01}}{x_{0\gamma}x_{1\gamma}} \right)^{2\gamma} \frac{1}{x_{\beta\gamma}^{2(1+\gamma_a-\gamma_b-\gamma)}} \frac{1}{x_{\gamma\alpha}^{2(1+\gamma_b-\gamma_a-\gamma)}}, \quad (2.9)$$

where the function f –the ‘triple Pomeron vertex’– can be found in [25, 26], and we have already set $x_{a_1} = x_{b_1} \equiv x_c$.

To make progress we assume that $\gamma_a = \gamma_b$, which is a good approximation when the configuration of the two child dipoles is more or less symmetric. (The saddle points γ_a and γ_b depend only logarithmically on dipole sizes.) Then the x_γ integral can be done [27]

$$\frac{1}{x_{\alpha\beta}^{2(1-\gamma)}} \int d^2x_\gamma \left(\frac{x_{01}}{x_{0\gamma}x_{1\gamma}} \right)^{2\gamma} \left(\frac{x_{\alpha\beta}}{x_{\alpha\gamma}x_{\beta\gamma}} \right)^{2(1-\gamma)} = \frac{1}{x_{\alpha\beta}^{2(1-\gamma)}} \left(c_\gamma |\rho|^{2\gamma} |F(\gamma, \gamma, 2\gamma, \rho)|^2 \right. \\ \left. + (\gamma \leftrightarrow 1 - \gamma) \right), \quad (2.10)$$

where F is the hypergeometric function,

$$c_\gamma = \pi 2^{-4i\nu-1} \frac{\Gamma(\frac{1}{2} + i\nu)\Gamma(-i\nu)}{\Gamma(\frac{1}{2} - i\nu)\Gamma(1 + i\nu)}, \quad (2.11)$$

and

$$\rho \equiv \frac{z_{01}z_{\alpha\beta}}{z_{0\alpha}z_{1\beta}}, \quad (2.12)$$

is the anharmonic ratio of the four points $(x_0, x_1, x_\alpha, x_\beta)$ (z is the complex coordinate representation of x), see figure 2. The remaining integrals are difficult to perform in full generality. As in [13], we shall restrict ourselves to two limiting cases $x_{01} \rightarrow 0$ (small parents) and $x_{01} \rightarrow \infty$ (large parents). In both limits, $|\rho| \ll 1$, so we may approximate $F(\dots, \rho) \approx 1$. The two terms in (2.10) give equal contributions due to the symmetry $\gamma \rightarrow 1 - \gamma$. Taking this into account, we can write

$$I = 2c_\gamma f(\gamma, \gamma_a, \gamma_a) \int \frac{d^2x_\alpha d^2x_\beta}{x_{\alpha\beta}^4} \left(\frac{x_{a_0c}x_{\alpha\beta}}{x_{a_0\alpha}x_{c\beta}} \right)^{2\gamma_a} \left(\frac{x_{b_0c}x_{\alpha\beta}}{x_{b_0\beta}x_{c\alpha}} \right)^{2\gamma_a} \left(\frac{x_{01}x_{\alpha\beta}}{x_{0\alpha}x_{1\beta}} \right)^{2\gamma}. \quad (2.13)$$

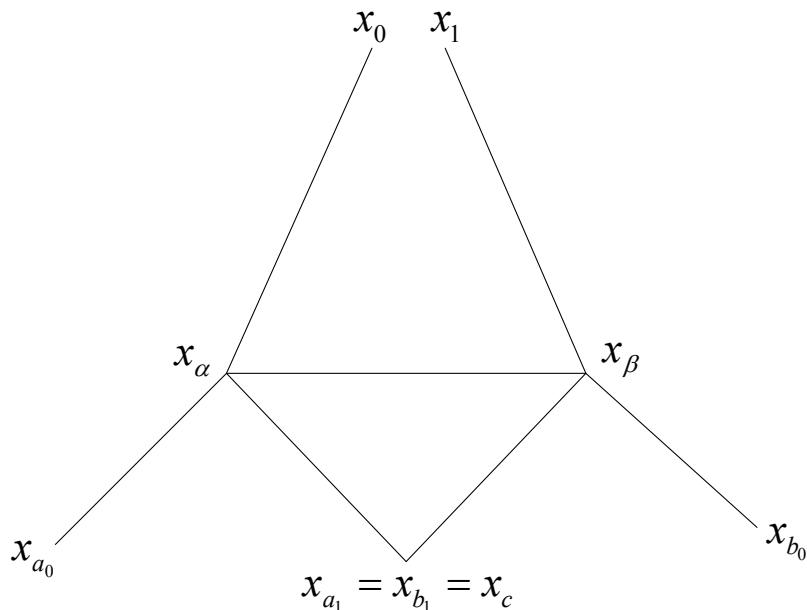


Figure 2: Equation (2.1) after integrating over x_1, x_2, x_3 and x_γ .

The integrand is a product of anharmonic ratios weighted by the conformally invariant measure $d^2x_\alpha d^2x_\beta/x_{\alpha\beta}^4$, so it is invariant under conformal transformations of the external points. However, since there are five of them ($x_0, x_1, x_{a_0}, x_{b_0}$ and x_c), conformal symmetry is not strong enough to constrain the solution, and our assumption $x_{01} \rightarrow \infty$ or $x_{01} \rightarrow 0$ will be crucial in the following.

2.2.1 Large parents

Suppose the parent dipole is large and the points $x_{a_0, b_0, c}$ are all located near the center of the parent dipole as illustrated in figure 3(a). This may be regarded as a situation relevant to DIS on a hadron at high photon virtuality. Without loss of generality, we can set $x_c = 0$. The integrand vanishes as $x_{\alpha, \beta} \rightarrow \infty$ very fast, so that a finite region of $x_{\alpha, \beta}$ near the origin is important. Therefore we may approximate

$$\frac{x_{01}}{x_{0\alpha}x_{1\beta}} \rightarrow \frac{4}{x_{01}}. \tag{2.14}$$

Under this assumption, (2.13) takes the form

$$I = 2c_\gamma f(\gamma, \gamma_a, \gamma_b) \left(\frac{4}{x_{01}}\right)^{2\gamma} \int \frac{d^2x_\alpha d^2x_\beta}{x_{\alpha\beta}^{4-2\gamma-4\gamma_a}} \left(\frac{x_{a_0}}{x_{a_0\alpha}x_\beta}\right)^{2\gamma_a} \left(\frac{x_{b_0}}{x_{b_0\beta}x_\alpha}\right)^{2\gamma_b}. \tag{2.15}$$

For simplicity, we assume that the two dipoles have the same size: $x_{a_0c} = x_{b_0c} = r$. (The region $x_{a_0c} \gg x_{b_0c}$ or $x_{a_0c} \ll x_{b_0c}$ gives a subleading contribution in the BFKL or the BK equation, see section 2.3.) Writing $z_{a_0} = re^{i\theta_a}$ and $z_{b_0} = re^{i\theta_b}$ and rescaling $x_{\alpha, \beta} \rightarrow rx_{\alpha, \beta}$

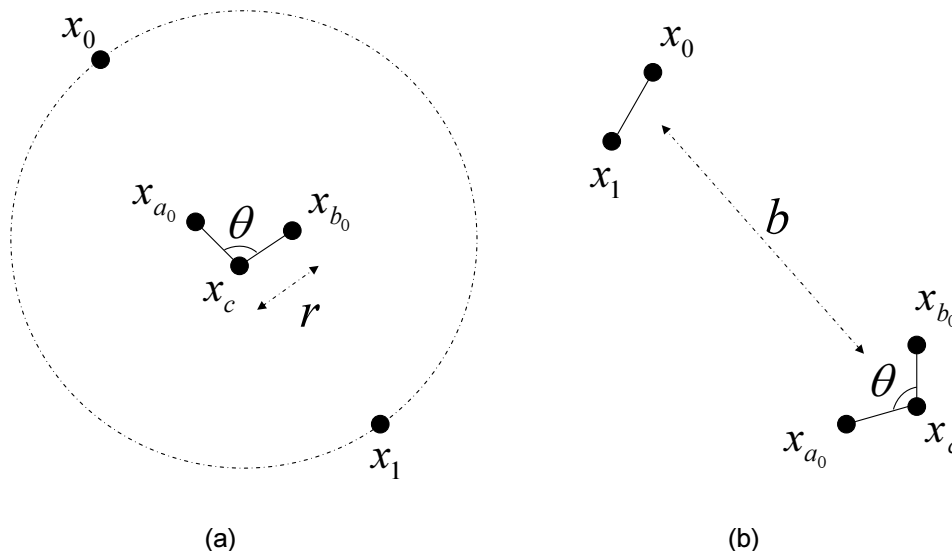


Figure 3: Graphical representation of the 'BK configurations', (a) for the large parent, small impact parameter case, and (b) for the small parent, large impact parameter case.

we get

$$\begin{aligned}
 I &= 2c_\gamma f(\gamma, \gamma_a, \gamma_b) \left(\frac{4r}{x_{01}}\right)^{2\gamma} \int \frac{d^2x_\alpha d^2x_\beta}{x_{\alpha\beta}^{4-2\gamma-4\gamma_a}} \left(\frac{1}{|e^{i\theta_a} - z_\alpha|x_\beta}\right)^{2\gamma_a} \left(\frac{1}{|e^{i\theta_b} - z_\beta|x_\alpha}\right)^{2\gamma_b} \\
 &\equiv 2c_\gamma f(\gamma, \gamma_a, \gamma_b) \left(\frac{4r}{x_{01}}\right)^{2\gamma} g(\theta), \tag{2.16}
 \end{aligned}$$

where $\theta = \theta_a - \theta_b$ is the relative angle between the two child dipoles. We have not been able to determine the function $g(\theta)$ for $\theta \neq 0$ in a closed form ($g(0)$ is a known integral in the conformal field theory literature [28–30]). But since this function has no singularity and depends only on the angle, it will not affect the evaluation of the saddle point below.

Neglecting this angular dependence and other prefactors, we can estimate the two dipole scattering amplitude as

$$\begin{aligned}
 T^{(2)}(x_{01}, x_{a_0c}, x_{b_0c}) &\sim \alpha_s^2 x_{a_0c}^2 x_{b_0c}^2 n^{(2)}(x_{01}, x_{a_0c}, x_{b_0c}) \\
 &\sim \alpha_s^2 \int_0^Y dy \int d\gamma d\gamma_a d\gamma_b \left(\frac{r}{x_{01}}\right)^{2\gamma} e^{\chi(\gamma)y + (\chi(\gamma_a) + \chi(\gamma_b))(Y-y)}. \tag{2.17}
 \end{aligned}$$

After performing the y integral, we get the two contributions

$$\int d\gamma d\gamma_a d\gamma_b \left(\frac{r}{x_{01}}\right)^{2\gamma} \frac{e^{\chi(\gamma)Y}}{\chi(\gamma) - \chi(\gamma_a) - \chi(\gamma_b)}, \tag{2.18}$$

and

$$\int d\gamma d\gamma_a d\gamma_b \left(\frac{r}{x_{01}}\right)^{2\gamma} \frac{e^{(\chi(\gamma_a) + \chi(\gamma_b))Y}}{\chi(\gamma_a) + \chi(\gamma_b) - \chi(\gamma)}. \tag{2.19}$$

The saddle point for the γ_a and γ_b integrals in (2.19) is simply the BFKL one $\gamma_a = \gamma_b = 1/2$, leading to

$$\int d\gamma \left(\frac{r}{x_{01}}\right)^{2\gamma} \frac{e^{2\chi(1/2)Y}}{2\chi(1/2) - \chi(\gamma)} \sim \left(\frac{r}{x_{01}}\right)^{2\gamma} e^{2\chi(1/2)Y}, \quad (2.20)$$

where γ solves

$$\chi(\gamma) = 2\chi\left(\frac{1}{2}\right), \quad \gamma \approx 0.82. \quad (2.21)$$

For the contribution (2.18) we can use the saddle point for the γ integral,

$$\chi'(\gamma_s)Y = \ln \frac{x_{01}^2}{r^2}, \quad (2.22)$$

and the leading rapidity behavior of this contribution is then given by

$$\left(\frac{r}{x_{01}}\right)^{2\gamma_s} e^{\chi(\gamma_s)Y}. \quad (2.23)$$

As we discuss in section 2.3, it holds that $2\chi(1/2) > \chi(\gamma_s)$, i.e., $\gamma_s < 0.82$ for all configurations we are interested in. (In the limit $Y \rightarrow \infty$, $\gamma_s \rightarrow 1/2$.) The contribution which dominates is thus given by (2.20), and we therefore have

$$T^{(2)} \sim \alpha_s^2 \left(\frac{r}{x_{01}}\right)^{2\gamma} e^{2\chi(1/2)Y}. \quad (2.24)$$

On the other hand, the single dipole scattering amplitude is given by

$$T(x_{01}, r) \sim \alpha_s \left(\frac{r}{x_{01}}\right)^{2\tilde{\gamma}} e^{\chi(\tilde{\gamma})Y}, \quad (2.25)$$

where $\tilde{\gamma}$ is the solution to

$$\chi'(\tilde{\gamma})Y = \ln \frac{x_{01}^2}{r^2}. \quad (2.26)$$

Taking the ratio, we arrive at

$$R \equiv \frac{T^{(2)}}{(T)^2} \sim \left(\frac{x_{01}}{r}\right)^{2(2\tilde{\gamma}-\gamma)} e^{2(\chi(1/2)-\chi(\tilde{\gamma}))Y}. \quad (2.27)$$

Since $2\tilde{\gamma} > 1 > \gamma$, the first factor is larger than 1 and predicts that the correlation increases as the asymmetry becomes larger $x_{01} \gg r$. Since $\chi(\tilde{\gamma}) > \chi(1/2)$, the second, exponential factor tends to decrease the correlation at high values of rapidity. Comparing this with (2.7), we infer that R monotonously increases and eventually saturates to the expression (2.27) as $x_{ab} \rightarrow 0$.

2.2.2 Small parents

Another tractable example is the limit of a small parent dipole $x_{01} \rightarrow 0$. In this case we may approximate $x_{1\beta} \approx x_{0\beta}$, after which the point x_1 drops out from the integral. Rewriting

$$I = 2c_\gamma f(\gamma, \gamma_a, \gamma_a) \left(\frac{x_{01}x_{a_0b_0}}{x_{0a_0}x_{0b_0}} \right)^{2\gamma} \int \frac{d^2x_\alpha d^2x_\beta}{x_{\alpha\beta}^4} \left(\frac{x_{a_0c}x_{b_0c}x_{\alpha\beta}^2}{x_{a_0\alpha}x_{c\alpha}x_{b_0\beta}x_{c\beta}} \right)^{2\gamma_a} \left(\frac{x_{\alpha\beta}x_{0a_0}x_{0b_0}}{x_{0\alpha}x_{0\beta}x_{a_0b_0}} \right)^{2\gamma} \quad (2.28)$$

we see that, apart from the prefactor, the integrand is conformally invariant, so it can be written as

$$I = 2c_\gamma f(\gamma, \gamma_a, \gamma_a) \left(\frac{x_{01}x_{a_0b_0}}{x_{0a_0}x_{0b_0}} \right)^{2\gamma} h(\eta, \eta^*), \quad (2.29)$$

where η is an anharmonic ratio

$$\eta \equiv \frac{z_{a_00}z_{b_0c}}{z_{a_0c}z_{b_00}}. \quad (2.30)$$

In order to evaluate the function h , one can set, using a conformal transformation, $x_0 = \infty$, $x_c = 0$, $x_{a_0} = 1$

$$h(z_{b_0}, \bar{z}_{b_0}) = \frac{x_{b_0}^{2\gamma_a}}{(1-x_{b_0})^{2\gamma}} \int d^2x_\alpha d^2x_\beta (x_\alpha - x_\beta)^{4\gamma_a+2\gamma-4} (1-x_\alpha)^{-2\gamma_a} \times (x_{b_0} - x_\beta)^{-2\gamma_a} x_\alpha^{-2\gamma_a} x_\beta^{-2\gamma_a}, \quad (2.31)$$

and therefore,

$$\begin{aligned} h(\eta, \bar{\eta}) &= \frac{|\eta|^{2\gamma_a}}{|1-\eta|^{2\gamma}} \int d^2x_\alpha d^2x_\beta (x_\alpha - x_\beta)^{4\gamma_a+2\gamma-4} (1-x_\alpha)^{-2\gamma_a} |\eta - z_\beta|^{-2\gamma_a} x_\alpha^{-2\gamma_a} x_\beta^{-2\gamma_a} \\ &= \left(\frac{x_{01}x_{a_0c}}{x_{0a_0}x_{0c}} \right)^{2\gamma} \left(\frac{x_{b_0c}x_{a_00}}{x_{a_0c}x_{0b_0}} \right)^{2\gamma_a} \int d^2x_\alpha d^2x_\beta \dots \end{aligned} \quad (2.32)$$

Remarkably, the same integral as in (2.15) appears, as a consequence of the symmetry between the limits $x_{01} \rightarrow \infty$ and $x_{01} \rightarrow 0$ found in [13]. First consider the case of large impact parameters $b \equiv |x_{0a_0}| \approx |x_{0b_0}| \approx |x_{0c}| \gg r$ (see, figure 3(b) and related calculations in [13, 31]). Then η is approximately a phase $\eta \approx e^{i\theta}$ where θ is the relative angle as before. We find³

$$I \approx 2c_\gamma f(\gamma, \gamma_a, \gamma_a) \left(\frac{x_{01}r}{b^2} \right)^{2\gamma} g(\theta), \quad (2.34)$$

³In fact, this result can be reached from (2.16) via a conformal transformation thanks to the conformal invariance of the original integral (2.13). Consider a $SL(2, \mathbb{C})$ transformation

$$z \rightarrow z' = \frac{-1}{z - 1/b}. \quad (2.33)$$

Under this, one has $x_{01} \rightarrow x'_{01} \approx x_{01}/x_0x_1 \approx 4/x_{01}$, $x_c = 0 \rightarrow x'_c = b$, $x_{a_0} = r \rightarrow x'_{a_0} = b/(1-br)$, and $x_{a_0c} = r \rightarrow r' \approx b^2r$. Therefore, $4r/x_{01} = x'_{01}r'/b^2$ as expected. Note finally that by definition a conformal transformation does not change the angle θ .

and

$$T^{(2)} \sim \alpha_s^2 \int_0^Y dy \int d\gamma d\gamma_a d\gamma_b \left(\frac{x_{01}r}{b^2}\right)^{2\gamma} e^{\chi(\gamma)y + (\chi(\gamma_a) + \chi(\gamma_b))(Y-y)}. \quad (2.35)$$

Again, the saddle points are given by $\gamma_a = \gamma_b = 1/2$, and we have the pole at $\chi(\gamma) = 2\chi(1/2)$. On the other hand, the single scattering amplitude at large impact parameter is

$$T(x_{01}, r, b) \sim \alpha_s \left(\frac{x_{01}r}{b^2}\right)^{2\tilde{\gamma}} e^{\chi(\tilde{\gamma})Y}, \quad (2.36)$$

where $\tilde{\gamma}$ is the solution to

$$\chi'(\tilde{\gamma})Y = \ln \frac{b^4}{x_{01}^2 r^2}. \quad (2.37)$$

Taking the ratio, we find

$$R = \frac{T^{(2)}}{(T)^2} \sim \left(\frac{b^2}{x_{01}r}\right)^{2(2\tilde{\gamma}-\gamma)} e^{2(\chi(1/2)-\chi(\tilde{\gamma}))Y}. \quad (2.38)$$

So in this case the correlation R decreases as either x_{01} or r (or both) is increased (keeping $x_{01}, r \ll b$).

In order to exhibit a symmetry with respect to the large dipole case, let us look at the case of small impact parameters, typically, $b \sim r \gg x_{01}$. We find

$$I \sim \left(\frac{x_{01}}{r}\right)^{2\gamma}, \quad (2.39)$$

while

$$T(x_{01}, r, b) \sim \alpha_s \left(\frac{x_{01}}{r}\right)^{2\tilde{\gamma}} e^{\chi(\tilde{\gamma})Y}, \quad (2.40)$$

with γ determined from

$$\chi'(\tilde{\gamma})Y = \ln \frac{r^2}{x_{01}^2}, \quad (2.41)$$

so that

$$R \sim \left(\frac{r}{x_{01}}\right)^{2(2\tilde{\gamma}-\gamma)} e^{2(\chi(1/2)-\chi(\tilde{\gamma}))Y}. \quad (2.42)$$

Compare with (2.27). As x_{01} increases, while keeping $x_{01} \ll r$, the correlation decreases. From the limiting behaviors, (2.27) ($x_{01} \gg r$) and (2.42) ($x_{01} \ll r$), we see that R is enhanced when the asymmetry (x_{01} vs. r) is large, and it presumably takes a minimum value around $x_{01} \sim r$.

2.3 Estimates and comments

Regarding the rapidity dependence, we note that $\tilde{\gamma} \rightarrow 1/2$ as $Y \rightarrow \infty$. Thus for large Y , the coefficient multiplying Y in the exponent in (2.27) and (2.42) tends to zero. For a fixed Y , this coefficient again tends to zero when $x_{01} \rightarrow r$, as can be seen from (2.26) and (2.41). Therefore the results (2.27) and (2.42) predict that the correlation R decreases faster with Y when $x_{01}/r \gg 1$ and $x_{01}/r \ll 1$, while if we extrapolate our results towards the symmetric limit $x_{01} \approx r$, we see that R is almost constant in Y .

From (2.26) we can guess that $\tilde{\gamma}$ is quite close to $1/2$. Let us therefore set $\tilde{\gamma} = 1/2 + \epsilon$ and expand the BFKL eigenfunction to linear order in ϵ . One then finds that

$$\epsilon \approx -\frac{1}{\psi''(1/2)\bar{\alpha}_s Y} \ln \frac{x_{01}}{r} = \frac{1}{14\zeta(3)\bar{\alpha}_s Y} \ln \frac{x_{01}}{r}, \quad (2.43)$$

where $\zeta(3) \approx 1.2$. If $x_{01}/r = 2$ we then find, for $\bar{\alpha}_s = 0.2$, $\epsilon \approx 0.21/Y$, and thus for $Y = 8$ we have $\epsilon \approx 0.03$, while for $Y = 12$ we find $\epsilon \approx 0.02$. For $2(2\tilde{\gamma} - \gamma)$ we then find the values 0.46 and 0.43 for $Y = 8$ and 12 respectively. If instead $x_{01}/r = 40$ we find $\epsilon \approx 0.14$, $2(2\tilde{\gamma} - \gamma) \approx 0.91$ and $\epsilon \approx 0.09$, $2(2\tilde{\gamma} - \gamma) \approx 0.73$ for $Y = 8$ and 12 respectively. For this values of $\tilde{\gamma}$ we also note that the exponent multiplying Y in (2.27) is quite small, for $\tilde{\gamma} = 0.64$ it is 0.14 while for $\tilde{\gamma} = 0.59$ it is 0.06 (all these estimates are valid for $\bar{\alpha}_s = 0.2$).

Thus if, for a fixed Y , we try to fit R as a function of x_{01}/r using a single effective power, ω , we would expect this fit to give a too strong increase close to the minimum, $x_{01}/r \sim 1$, whereas it should give a too slow increase further away from the minimum. As $2(2\tilde{\gamma} - \gamma)$ varies stronger for smaller Y , we would expect the fit to work better for higher Y . We would also expect ω to be larger for smaller Y .

In the next section we will see that these analytical estimates are all in quite good agreement with the numerical results. In particular, the numerical analysis will confirm that the minimum of R (for zero impact parameter) occurs at $x_{01} \approx r$. Moreover, the estimates for $\tilde{\gamma}$ given above agree very well with the numerical results, and also the Y dependence turns out to be correct.

Before moving on to the numerical analysis, we would like to address one more point. So far we have been able to make analytic estimates only for specific configurations. In particular, we assumed that the dipoles x_{a_0c} and x_{b_0c} are more or less equal in size. In going from (1.2) to (1.1), however, the question is whether the replacement

$$\int d^2z \mathcal{M}(x, y, z) \cdot T_Y^{(2)}(x, z; z, y) \rightarrow \int d^2z \mathcal{M}(x, y, z) \cdot T_Y(x, z)T_Y(z, y), \quad (2.44)$$

is valid. (We have here returned to the notation used in the introduction using x, y and z .) What we have shown above is that $T^{(2)}(x, z; z, y) \gg T(x, z)T(z, y)$ for some specific regions of z , and also for specific relations between (x, y) and the target, but this is not sufficient to see the integrated effect of the correlation. Although one can use the MC code to do the integration over z , this can be quite time consuming. Leaving the numerical integration for future work, we here crudely identify the configurations which dominate the integral in (1.1). Consider the large parent case where $|x - y| \ll x_{01}$ and assume that $|x - y|$ is smaller than the saturation length Q_s^{-1} . This means that we may set $T(x, y) = (x - y)^2 Q_s^2$.

(We could also introduce an anomalous dimension $\gamma \neq 1$ but this is not essential.) We then divide the integral into three regions:

- Region A: $|x - z|, |y - z| \lesssim |x - y|$.
- Region B: $|x - y| \lesssim |x - z| \approx |z - y| \lesssim Q_s^{-1}$.
- Region C: $Q_s^{-1} \lesssim |x - z| \approx |z - y|$.

In region A we have

$$\int_A d^2 z \frac{(x - y)^2}{(x - z)^2 (y - z)^2} \times \left\{ (x - z)^2 Q_s^2 + (z - y)^2 Q_s^2 - (x - y)^2 Q_s^2 - (x - z)^2 Q_s^2 \cdot (x - y)^2 Q_s^2 \right\} \sim (x - y)^2 Q_s^2. \quad (2.45)$$

(Note that there is no logarithmic singularity at either $z = x$ or $z = y$.) In region B we instead have

$$\int_B d^2 z \frac{(x - y)^2}{(x - z)^4} \left\{ -(x - y)^2 Q_s^2 + 2(x - z)^2 Q_s^2 - (x - z)^4 Q_s^4 \right\} \approx (x - y)^2 \int_B d^2 z \frac{1}{(x - z)^4} 2(x - z)^2 Q_s^2 \sim (x - y)^2 Q_s^2 \ln \frac{1}{(x - y)^2 Q_s^2}, \quad (2.46)$$

while in region C we have

$$\int_C d^2 z \frac{(x - y)^2}{z^4} \left\{ -(x - y)^2 Q_s^2 + 1 \right\} \sim (x - y)^2 Q_s^2, \quad (2.47)$$

where the integral is dominated by the lower limit $|x - z| \sim 1/Q_s$. Thus for a small projectile which has not yet reached saturation $|x - y| \ll Q_s^{-1}$, the dominant contribution comes from region B where we indeed have $|x - z| \approx |z - y|$. As $|x - y| \rightarrow 1/Q_s$, region B shrinks, and the dominant region is simply $|x - z| \sim |z - y| \sim |x - y|$. Therefore, we expect that the configurations we are using are relevant, and the large correlation found there should survive after integrating over z in the evolution equation.

3. Numerical Approach

3.1 Outline of the approach

In this section we will perform a numerical analysis to compute the quantities $T^{(2)}$ and $(T)^2$. This can be done rather easily in a Monte Carlo implementation of the dipole model, and we will here use the C++ code developed in [18]. The calculation we will perform is straightforward, no matter which configuration we have. Recall that the definitions of T

and $T^{(2)}$ are

$$T_Y(x, y) = \int d^2u d^2v A_0(x, y|u, v) n_Y(u, v), \quad (3.1)$$

$$\begin{aligned} T_Y^{(2)}(x_1, y_1; x_2, y_2) &= \int d^2u_1 d^2v_1 d^2u_2 d^2v_2 A_0(x_1, y_1|u_1, v_1) \times \\ &\quad \times A_0(x_2, y_2|u_2, v_2) n_Y^{(2)}(u_1, v_1; u_2, v_2) \\ &\quad + \int d^2u d^2v A_0(x_1, y_1|u, v) A_0(x_2, y_2|u, v) n_Y(u, v), \end{aligned} \quad (3.2)$$

where A_0 is the elementary dipole-dipole scattering amplitude. (The second term on the right hand side of (3.1) represents scattering of two dipoles off the same dipole in the target.) Starting from any initial dipole distribution, the MC code evolves the initial state up to a given value of Y , after which one can calculate all possible scatterings between the dipoles. The Monte Carlo estimate of equation (3.1) is simply given by

$$T_{MC}^{(2)}(x_1, y_1; x_2, y_2) = \frac{1}{N_{ev}} \sum_{n=1}^{N_{ev}} \sum_{i,j \in \Gamma_n} A_0(x_1, y_1|u_i, v_i) \cdot A_0(x_2, y_2|u_j, v_j), \quad (3.3)$$

where Γ_n is the configuration of the evolved target for the n th event. Writing $\sum_{i,j} = \sum_{i \neq j} + \sum_i$ we see that (3.3) contains both contributions in (3.1). In writing this formula we only evolved the target but we can obviously do the computation in any given frame. Similarly the product $T(x_1, y_1) \cdot T(x_2, y_2)$ is calculated as

$$T_{MC}(x_1, y_1) \cdot T_{MC}(x_2, y_2) = \frac{1}{N_{ev}} \sum_{n=1}^{N_{ev}} \sum_{i \in \Gamma_n} A_0(x_1, y_1|u_i, v_i) \cdot \frac{1}{N_{ev}} \sum_{n=1}^{N_{ev}} \sum_{i \in \Gamma_n} A_0(x_2, y_2|u_i, v_i). \quad (3.4)$$

In the next section we will start by checking the predictions from [13] as stated in equations (1.4) and (2.7). As in the analytical approach we consider a target which initially consists of a single dipole (x_0, x_1) (for the numerical calculation we could start from any configuration if we so wish) For the configurations in [13], the phenomenologically more relevant configuration is the one in which the target x_{01} is much larger than the projectile dipoles. We fix the projectile dipoles to have the same size, $r = x_{a_0 a_1} = x_{b_0 b_1}$ (for the above formulas this means we have $x_1 = x_{a_0}, y_1 = x_{a_1}, x_2 = x_{b_0}, y_2 = x_{b_1}$), while the distance between them, x_{ab} , will be varied.

For the BK configurations, we have $x_2 = y_1 = x_c$, and again we fix the two projectile dipoles to have the same size, $r = |x_{a_0} - x_c| = |x_c - x_{b_0}|$. The target dipole (x_0, x_1) is placed at zero impact parameter, as in figure 3 (a), while its orientation is chosen randomly for each event. We will always keep x_{a_0}, x_{b_0} and x_c fixed while we vary x_{01} and the impact parameter.

One technical point is that one has to introduce a cutoff, ρ , for the minimal size of dipoles generated during the evolution since the dipole kernel $\mathcal{M}(x, y, z)$ diverges at $z = x$ and $z = y$. Such a cutoff explicitly breaks conformal symmetry, and one should therefore ideally choose a cutoff which is much smaller than the relevant scales (the initial dipole sizes) involved in the process. On the other hand, simulations with too small values of ρ

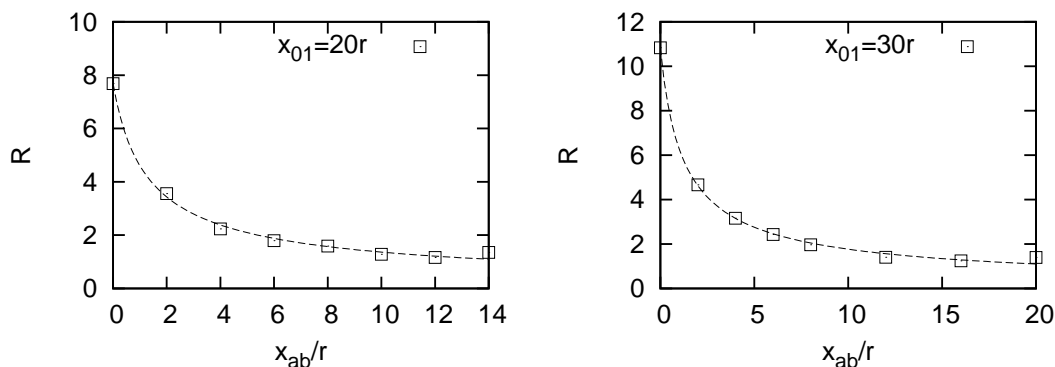


Figure 4: The numerical results for the configurations described in (1.4) at $Y = 10$, and for target of size $x_{01} = 20r$ (left plot) and $x_{01} = 30r$ (right plot). The MC results are shown as squares while the power-like fits to the results are shown as dashed lines.

are very time-consuming. If one is studying symmetric collisions $r \sim x_{01}$, then the choice $\rho = 0.01r = 0.01x_{01}$ is good enough. Choosing an even smaller ρ in this case is not useful since one is then wasting a lot of time to generate many very small dipoles which do not interact and do not contribute much to the scattering amplitude. However, here we wish to study the correlation as we vary x_{01} , and then the choice of ρ is more subtle. For example, for a very asymmetric collision, say $x_{01} \sim 100r$, ρ has to be much smaller than $0.01x_{01}$ so that we do not suppress important dipoles with size of order r . Besides, in the absence of saturation effects, smallness of ρ is also required for the frame-independence of $T^{(2)}$, hence that of R . As a compromise between these requirements (reducing simulation time and ensuring frame-independence) we shall choose $\rho(x_{01}) = 0.05r$ throughout. With this choice we confirmed that the results presented in what follows are reasonably frame-independent even up to the center-of-mass frame.

3.2 Results

As mentioned above we start by checking the results from [13]. The target will be fixed at the origin, with random orientation, and the projectile dipoles are placed symmetrically along the horizontal axis, one on the positive axis and the other on the negative axis, with random orientations. We choose $\bar{\alpha}_s = 0.2$ throughout, except in the running coupling case to be presented later.

The results for this configuration are shown in figure 4. Here we choose $x_{01} = 20r$ in the left plot, and $x_{01} = 30r$ in the right plot keeping $x_{01} > x_{ab}$. The former case would in DIS correspond to a virtuality of $Q^2 \sim 60 \text{ GeV}^2$. In both cases we also show fits of the form $R = \alpha/(x_{ab} + \beta)^\gamma$. We thus confirm the power-like behavior in (1.4), and also see that R converges to a finite value as $x_{ab} \rightarrow 0$ in agreement with the analytical prediction (2.27). For the left plot the fit gives the values $\beta = 0.09$ and $\gamma = 0.70$ while for the right plot we get $\beta = 0.09$ and $\gamma = 0.72$.

Next we turn to the BK configuration described in the previous section. In figure 5, we plot R as a function of x_{01}/r for $Y = 6, 8$ and 10 , at zero impact parameter. We can see a behavior of R consistent with the analytical formulas, equations (2.27) and (2.42). The

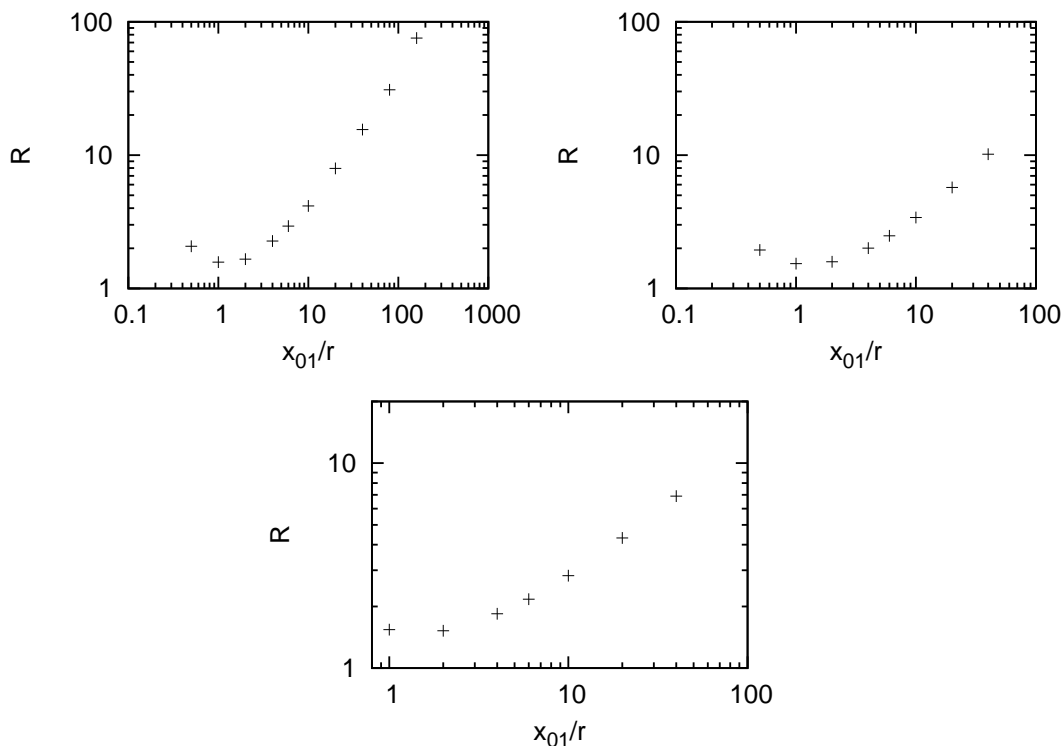


Figure 5: The numerical results for R at $Y = 6$ (upper left plot), $Y = 8$ (upper right plot) and $Y = 10$ (bottom plot) at zero impact parameter.

minimum of R indeed occurs at $x_{01} \approx r$ with the minimal value $R \approx 1.5$. For asymmetric configurations, R can easily reach values of order 10. Moreover, the powers extracted from figure 5 agree with the expectations from equations (2.21) and (2.26). For $Y = 6$, a fit of the form $(x_{01}/r)^\omega$ gives the values $\omega = 0.55$ in the region $x_{01}/r = 1 \rightarrow 10$, $\omega = 0.93$ in the region $x_{01}/r = 6 \rightarrow 40$, and $\omega = 1.22$ in the region $x_{01}/r = 40 \rightarrow 200$. If we instead calculate the power by calculating $\tilde{\gamma}$ using equation (2.43) at the points $x_{01}/r = 5, 20$ and 120 representing the three regions above, we find the respective values $0.68, 0.96$ and 1.32 , in very good agreement with the numerical results. Similarly, for $Y = 8$ we find the values $\omega = 0.52$ and 0.76 from fits in the first two regions above. This can be compared to the analytical result which gives $\omega = 0.60$ and 0.80 .

From our analysis in the previous section we know that R decreases as Y increases, and the rate of decrease is larger for asymmetric scattering. This tendency can be clearly observed, though the ratio R doggedly stays $\gtrsim 1.5$. In the current simulation we cannot go to larger values of Y because the single dipole amplitude T for $x_{01} \sim r$ reaches order unity around $Y = 10$. Therefore, in the entire domain of Y values where our approach makes sense, the mean field approximation $R = 1$ is nowhere valid even in central collisions. Since this persists up to the onset of the strong scattering regime $T \sim \mathcal{O}(1)$, it is unlikely that saturation effects immediately wash out the correlation. Rather, one has to carefully study the effect of correlations when solving nonlinear equations.

Another, perhaps more striking consequence of the correlation emerging from our anal-

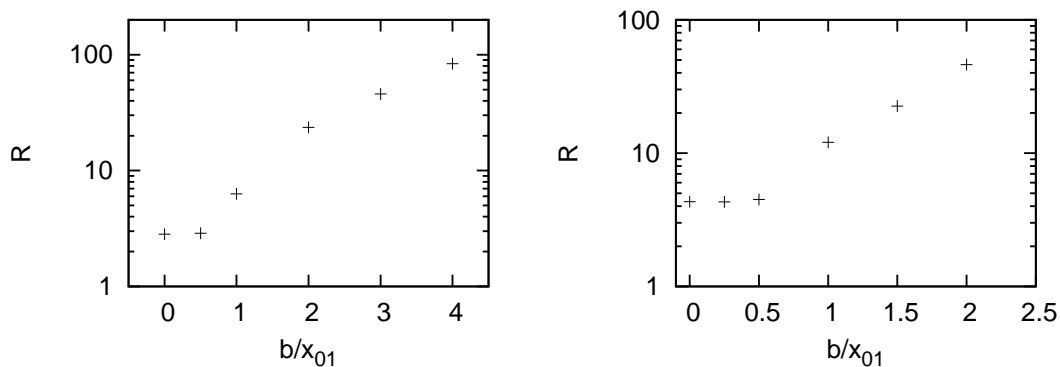


Figure 6: The numerical results for R at nonzero impact parameter $b \neq 0$, $Y = 10$ and $x_{01} = 10r$ (left plot) and $x_{01} = 20r$ (right plot).

ysis is that it makes the nonlinear term $T^{(2)}$ comparable to T even when $T \ll 1$. For example, we have $T = 0.023$ for $x_{01} = 40r$ at $Y = 8$, and in this case we see from figure 5 that $R = 10$. This means that $T^{(2)} = 0.0056$, and thus $T^{(2)}/T = 0.24$, so $T^{(2)}$ is not completely negligible as compared with T . For the more symmetric case $x_{01} = 6r$ at $Y = 10$ we have $T = 0.39$ while $T^{(2)} = 0.32$ and $R = 2.2$, see again figure 5. For $x_{01} = 40r$ and $Y = 10$ we instead have $T = 0.059$, while $R = 6.9$ and therefore $T^{(2)}/T = 0.41$. Taken at face value, these estimates suggests that one might have to include the nonlinear effects in the evolution already in the dilute regime where $T \ll 1$. We did not include such a back-reaction into our linear dipole evolution, and in this regard our analysis is not complete. This point certainly deserves further study.

So far we have studied only configurations with zero impact parameter $b = 0$. At finite impact parameter the correlation becomes larger as suggested by (2.38). Of course if we think of x_{01} as representing the proton radius then one should be careful in interpreting results for $b \gg x_{01}$ where confinement effects are certainly important. As a check of the analytical prediction, and also for the sake of demonstration, we nevertheless present some results when $b > x_{01}$. Figure 6 shows the b dependence of R for $x_{01}/r = 10$ and $x_{01}/r = 20$. We see that R is almost constant as long as b is smaller than x_{01} and that it grows rapidly when $b \gtrsim x_{01}$.

Numerical simulation with a running coupling

One of the non-leading effects which we can easily incorporate into the numerical simulation is the running coupling as has already been done in [18–20]. Although in this paper we mainly concentrate ourselves on the fixed coupling case, we would here like to briefly mention some of the preliminary results obtained when the running coupling is used.

Technically, the inclusion of the running coupling is completely straightforward and we shall use the one-loop expression for α_s ,

$$\alpha_s(Q^2) = \frac{4\pi}{\left(\frac{11}{3}N_c - \frac{2}{3}n_f\right) \ln(Q^2/\Lambda_{\text{QCD}}^2)} \quad (3.5)$$

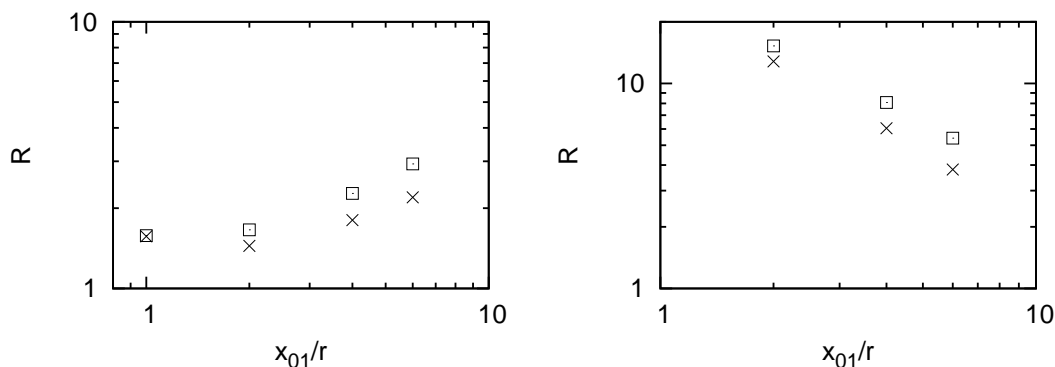


Figure 7: The crosses are the numerical estimates of R obtained using a running coupling at $Y = 6$, and for $b = 0$ (left plot) and $b = 5r$ (right plot). The squares are the corresponding fixed coupling results.

where we fix $\Lambda_{\text{QCD}} = 0.22\text{GeV}$. The running coupling enters both in the dipole evolution (as $\bar{\alpha}_s$) and in the individual dipole-dipole scatterings (as α_s^2). We will set $N_c = 3$ and $n_f = 3$ as in [19, 20].

To avoid the IR singularity we shall freeze the coupling below a minimum scale Q_{min} corresponding to a maximum dipole size $r_{\text{max}} = 1/Q_{\text{min}}$. As in [20], we choose $r_{\text{max}} = 3.5\text{GeV}^{-1}$. In [20], α_s was evaluated at the scale $1/Q = \min(r, r_1, r_2)$ for the splitting $r \rightarrow r_1, r_2$, and this choice roughly follows from next-to-leading log (NLL) studies of the dipole evolution [32, 33]. [See section VII of [34] for a compact discussion.] Thus we continue to use this scale in the evolution of the dipole cascade. For the dipole-dipole interaction the correct choice of the scale is more subtle, and we here use the option described in [20].

In practice, simulations with the running coupling are quite time-consuming, and we have therefore not been able to check as many configurations as in the fixed coupling case. In figure 7 we show the results obtained at $Y = 6$ both at zero (left plot) and nonzero (right plot) impact parameter, together with the fixed coupling results. We see that R is somewhat reduced, but its minimum value is still around 1.5. We also see that the qualitative behavior of R does not change, the minimum again occurs when $x_{01} \approx r$ although it is of course difficult to determine the exact behavior of R since we do not have enough data points. At $Y = 8$ for $b = 0$, we find the value $R = 1.5$ at $x_{01} = 2r$, while in the fixed coupling case we found $R = 1.6$. For $b = 5r$, R reduces from 11.6 in the fixed coupling case to 9.4 in the running coupling case for the same configuration.

4. Conclusions

In this paper we have studied both analytically and numerically the correlations induced by the leading order BFKL dynamics in the high energy evolution of a dilute system (such as a proton). Our main analytical results are given in equations (2.27), (2.38) and (2.42). All these results indicate that one should expect power-like correlations which lead to a strong violation of the factorization $T^{(2)} \approx T \cdot T$. The analytical estimates have been

demonstrated to be qualitatively correct by a numerical analysis with which we have also been able to quantitatively study the behavior of the ratio $R = T^{(2)}/T^2$. We have found that R is always larger than ~ 1.5 and it can easily reach $\sim \mathcal{O}(10)$ when the asymmetry is large.

Physical consequences of the correlation remain to be explored. The first and obvious intuition is that it opens an intriguing possibility of the ‘grey disc’ limit in which a scattering amplitude saturates to a value less than 1.⁴

$$T \rightarrow \frac{1}{R} < 1. \quad (4.1)$$

However, since R is not a constant, and the nonlinear equations involve an integration over the transverse plane with a nontrivial weight, a more detailed analysis would be required in order to draw any conclusions.

Another interesting problem is the interplay with the gluon number fluctuation which has attracted considerable attention lately (see [36] and references therein), but which has so far mostly been studied in simple toy models where the transverse dimensions are suppressed. Though it typically requires unrealistically large energies to see the impact of the gluon number fluctuation on the nonlinear evolution of large nuclei, this is probably not the case for a dilute target. The BFKL evolution generates a very strong number fluctuation as well as the transverse correlation in the dilute regime, and they can both affect the subsequent nonlinear evolution in significant ways.

There is plenty of room for improvements in the Monte Carlo simulation itself. In order to make a quantitative prediction for realistic experiments, one should include various NLL corrections and saturation effects into the target evolution. They have been incorporated in the dipole model in [18–20]. Among them, we have in this paper included some results with the running coupling effect. Since our simulations have been limited in size, it is difficult to determine the exact behavior of R . What we have clearly observed, however, is that R is somewhat reduced from the fixed coupling case, but is still large. This suggests that the large correlation may not be totally attributed to conformal symmetry of the leading order BFKL, but rather is a robust feature of the QCD evolution in the linear regime.

As mentioned in the introduction we would expect even larger correlations in the multiple scattering amplitudes $T^{(p)}$ ($p \geq 3$) which enter the Balitsky hierarchy. In the dipole model, these amplitudes are directly related to the corresponding multiple dipole distributions $n^{(p)}$ [23, 24], but analytical results for them are scarce [37]. The numerical evaluation of these amplitudes is straightforward, although the calculation of $T^{(p)}$ for large p would be time-consuming due to the need of good statistics.

Acknowledgments

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⁴Such a possibility was previously considered in [11, 35] in the context of nonlinear equations, although the parameter R in these works was fixed by some arguments unrelated to the BFKL evolution.

References

- [1] I. Balitsky, *Operator expansion for high-energy scattering*, *Nucl. Phys.* **B 463** (1996) 99 [[hep-ph/9509348](#)].
- [2] Y.V. Kovchegov, *Small- x F_2 structure function of a nucleus including multiple Pomeron exchanges*, *Phys. Rev.* **D 60** (1999) 034008 [[hep-ph/9901281](#)].
- [3] E.A. Kuraev, L.N. Lipatov and V.S. Fadin, *The Pomeron singularity in nonabelian gauge theories*, *Sov. Phys. JETP* **45** (1977) 199 [*Zh. Eksp. Teor. Fiz.* **72** (1977) 199].
- [4] I.I. Balitsky and L.N. Lipatov, *The Pomeron singularity in quantum chromodynamics*, *Sov. J. Nucl. Phys.* **28** (1978) 822 [*Yad. Fiz.* **28** (1978) 1597].
- [5] H. Weigert, *Evolution at small $x(bj)$: the color glass condensate*, *Prog. Part. Nucl. Phys.* **55** (2005) 461 [[hep-ph/0501087](#)].
- [6] J. Jalilian-Marian and Y.V. Kovchegov, *Saturation physics and deuteron gold collisions at RHIC*, *Prog. Part. Nucl. Phys.* **56** (2006) 104 [[hep-ph/0505052](#)].
- [7] J. Jalilian-Marian, A. Kovner, A. Leonidov and H. Weigert, *The Wilson renormalization group for low x physics: towards the high density regime*, *Phys. Rev.* **D 59** (1999) 014014 [[hep-ph/9706377](#)].
- [8] J. Jalilian-Marian, A. Kovner and H. Weigert, *The Wilson renormalization group for low x physics: gluon evolution at finite parton density*, *Phys. Rev.* **D 59** (1999) 014015 [[hep-ph/9709432](#)].
- [9] E. Iancu, A. Leonidov and L.D. McLerran, *Nonlinear gluon evolution in the color glass condensate. I*, *Nucl. Phys.* **A 692** (2001) 583 [[hep-ph/0011241](#)].
- [10] E. Ferreiro, E. Iancu, A. Leonidov and L. McLerran, *Nonlinear gluon evolution in the color glass condensate. II*, *Nucl. Phys.* **A 703** (2002) 489 [[hep-ph/0109115](#)].
- [11] E. Levin and M. Lublinsky, *A linear evolution for non-linear dynamics and correlations in realistic nuclei*, *Nucl. Phys.* **A 730** (2004) 191 [[hep-ph/0308279](#)].
- [12] K. Rummukainen and H. Weigert, *Universal features of JIMWLK and BK evolution at small x* , *Nucl. Phys.* **A 739** (2004) 183 [[hep-ph/0309306](#)].
- [13] Y. Hatta and A.H. Mueller, *Correlation of small- x gluons in impact parameter space*, *Nucl. Phys.* **A 789** (2007) 285 [[hep-ph/0702023](#)].
- [14] M. Braun and D. Treleani, *The double parton distributions in the hard Pomeron model*, *Eur. Phys. J.* **C 18** (2001) 511 [[hep-ph/0005078](#)].
- [15] A.H. Mueller, *Soft gluons in the infinite momentum wave function and the BFKL Pomeron*, *Nucl. Phys.* **B 415** (1994) 373.
- [16] G.P. Salam, *Studies of unitarity at small x using the dipole formulation*, *Nucl. Phys.* **B 461** (1996) 512 [[hep-ph/9509353](#)].
- [17] A.H. Mueller and G.P. Salam, *Large multiplicity fluctuations and saturation effects in onium collisions*, *Nucl. Phys.* **B 475** (1996) 293 [[hep-ph/9605302](#)].
- [18] E. Avsar, G. Gustafson and L. Lönnblad, *Energy conservation and saturation in small- x evolution*, *JHEP* **07** (2005) 062 [[hep-ph/0503181](#)].

- [19] E. Avsar, G. Gustafson and L. Lönnblad, *Small- x dipole evolution beyond the large- N_c limit*, *JHEP* **01** (2007) 012 [[hep-ph/0610157](#)].
- [20] E. Avsar, G. Gustafson and L. Lönnblad, *Diffractional excitation in DIS and pp collisions*, *JHEP* **12** (2007) 012 [[arXiv:0709.1368](#)].
- [21] A.H. Mueller and B. Patel, *Single and double BFKL Pomeron exchange and a dipole picture of high-energy hard processes*, *Nucl. Phys.* **B 425** (1994) 471 [[hep-ph/9403256](#)].
- [22] A.H. Mueller, *Unitarity and the BFKL Pomeron*, *Nucl. Phys.* **B 437** (1995) 107 [[hep-ph/9408245](#)].
- [23] R.B. Peschanski, *Dual Shapiro-Virasoro amplitudes in the QCD dipole picture*, *Phys. Lett.* **B 409** (1997) 491 [[hep-ph/9704342](#)].
- [24] M.A. Braun and G.P. Vacca, *Triple Pomeron vertex in the limit $N_c \rightarrow \infty$* , *Eur. Phys. J.* **C 6** (1999) 147 [[hep-ph/9711486](#)].
- [25] A. Bialas, H. Navelet and R.B. Peschanski, *The QCD triple Pomeron coupling from string amplitudes*, *Phys. Rev.* **D 57** (1998) 6585 [[hep-ph/9711442](#)].
- [26] G.P. Korchemsky, *Conformal bootstrap for the BFKL Pomeron*, *Nucl. Phys.* **B 550** (1999) 397 [[hep-ph/9711277](#)].
- [27] L.N. Lipatov, *Small- x physics in perturbative QCD*, *Phys. Rept.* **286** (1997) 131 [[hep-ph/9610276](#)].
- [28] V.S. Dotsenko and V.A. Fateev, *Conformal algebra and multipoint correlation functions in 2D statistical models*, *Nucl. Phys.* **B 240** (1984) 312.
- [29] R. Guida and N. Magnoli, *Tricritical Ising model near criticality*, *Int. J. Mod. Phys.* **A 13** (1998) 1145 [[hep-th/9612154](#)].
- [30] S. Bondarenko and A. Prygarin, *BFKL ansatz for BK equation in conformal basis*, *Nucl. Phys.* **A 800** (2008) 63 [[arXiv:0709.3010](#)].
- [31] A. Bialas and R.B. Peschanski, *Initial state color dipole emission associated with QCD Pomeron exchange*, *Phys. Lett.* **B 355** (1995) 301 [[hep-ph/9504293](#)].
- [32] I. Balitsky, *Quark contribution to the small- x evolution of color dipole*, *Phys. Rev.* **D 75** (2007) 014001 [[hep-ph/0609105](#)].
- [33] Y.V. Kovchegov and H. Weigert, *Triumvirate of running couplings in small- x evolution*, *Nucl. Phys.* **A 784** (2007) 188 [[hep-ph/0609090](#)].
- [34] I. Balitsky and G.A. Chirilli, *Next-to-leading order evolution of color dipoles*, *Phys. Rev.* **D 77** (2008) 014019 [[arXiv:0710.4330](#)].
- [35] R.A. Janik and R.B. Peschanski, *QCD saturation equations including dipole-dipole correlation*, *Phys. Rev.* **D 70** (2004) 094005 [[hep-ph/0407007](#)].
- [36] A. Dumitru, E. Iancu, L. Portugal, G. Soyez and D.N. Triantafyllopoulos, *Pomeron loop and running coupling effects in high energy QCD evolution*, *JHEP* **08** (2007) 062 [[arXiv:0706.2540](#)].
- [37] B.-W. Xiao, *On the anomalous dimensions of the multiple Pomeron exchanges*, *Nucl. Phys.* **A 798** (2008) 132 [[arXiv:0710.1922](#)].